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On an Elliptic Cone of Internal Refraction for Quasi-Transverse Waves in Tetragonal Crystals*

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Circular cones of internal and external refraction were shown to exist for light waves in optically biaxial crystals by Hamilton in 1833. Recently, a circular cone of internal refraction has been shown to exist for transverse elastic waves in a cubic crystal. An analysis is now presented which shows that an elliptic cone of internal refraction may exist for quasi-transverse waves of appropriate common normal propagated in a tetragonal crystal. Values of the elastic constants of zircon are used to illustrate the phenomenon.

Introduction

A general method for the investigation of the propagation of elastic waves in aeolotropic media (Musgrave, 1954) has already been used to predict the existence of a circular cone of internal refraction for transverse waves of normal (1, 1, 1) in a cubic crystal (de Klerk & Musgrave, 1955; Miller & Musgrave, 1956).

In this paper a similar analysis for a medium of simple tetragonal symmetry is presented. Such a medium possesses six independent elastic constants —c_{11}, c_{33}, c_{12}, c_{13}, c_{44}, c_{66}—and has reflexion symmetry about planes of normal (1, 0, 0), (0, 1, 0), (0, 0, 1) and $(1, \bar{1}, 0)$.

When a wave normal exists for which the phase velocities v_{T_1} and v_{T_2} of quasi-transverse waves are equal, there is an infinite set of possible displacement vectors associated with a single plane wave-front. If the corresponding rays are extraordinary, then a cone of refraction will exist; that is, tangency of the plane wavefront with the wave surface takes place at all points of a curve.

The following analysis shows that a wave normal of the form (m, m, n) in a tetragonal crystal may fulfil the required condition and that the curve of tangency is then an ellipse. Thus we may predict an elliptic cone of internal refraction for quasi-transverse waves.

Propagation of waves in a tetragonal crystal

The condition for non-zero displacement vectors for plane elastic waves of normal (l, m, n) in a medium of tetragonal symmetry and of density ϱ yields an equation for the possible phase velocities, v, which may be written

$$\begin{vmatrix} A-\varrho v^2, & \alpha\beta, & \gamma\alpha \\ \alpha\beta, & B-\varrho v^2, & \beta\gamma \\ \gamma\alpha, & \beta\gamma, & C-\varrho v^2 \end{vmatrix} = 0\,, \qquad (1)$$

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where

$$\left.\begin{aligned} A &= l^2a+m^2k+c_{44}, & \alpha^2 &= l^2f\,, \\ B &= l^2k+m^2a+c_{44}, & \beta^2 &= m^2f\,, \\ C &= n^2h+c_{44}, & \gamma^2 &= n^2d^2/f\,, \end{aligned}\right\} \qquad (2)$$

and

$$\left.\begin{aligned} a = c_{11}-c_{44},\ \ d = c_{13}+c_{44},\ \ f = c_{12}+c_{66}\,, \\ h = c_{33}-c_{44} \ \text{ and } \ k = c_{66}-c_{44}\,. \end{aligned}\right\} \qquad (3)$$

For wave normals of the type (m, m, n), equation (1) simplifies to

$$(H-m^2c)[H^2-\{m^2(c+2f)+n^2h\}H + m^2n^2\{h(c+2f)-2d^2\}] = 0\,, \qquad (4)$$

where

$$H = \varrho v^2-c_{44} \quad \text{and} \quad c = c_{11}-c_{12}-2c_{44}\,. \qquad (5)$$

Provided the roots of equation (4) are distinct, the root $H = m^2c$ is associated with a truly transverse (T_1) displacement vector $(1, \bar{1}, 0)$; the remaining roots H_i ($i = L$ or T_2) are in general associated with displacement vectors, one quasi-longitudinal and one quasi-transverse, and each of the form (q_i, q_i, r_i), where

$$q_i : r_i = \frac{mf}{H_i-m^2c} : \frac{nd}{H_i-n^2(h-d^2/f)}\,. \qquad (6)$$

A special case, however, arises if the velocities of the T_1 and T_2 waves are equal, that is if $H_{T_1} = H_{T_2}$. This may occur if

$$m^2n^2(fh-d^2)-m^4cf = 0\,,$$

which requires $m = 0$, when $H_{T_1} = 0 = H_{T_2}$,

or

$$m^2/n^2 = (fh-d^2)/cf\,, \qquad (7)$$

when $H_{T_1} = m^2c = H_{T_2}$.

Clearly condition (7) is fulfilled by real direction cosines only if

$$(fh-d^2)/cf > 0\,. \qquad (8)$$

Let us write m' and n' for the direction cosines which fulfil condition (7) and examine the behaviour of waves having a normal (m', m', n').

Plane waves of normal (m', m', n')

Corresponding to the largest root of equation (4), H_L, there is a displacement vector (q_L, q_L, r_L) such that

$$q_L : r_L = \frac{m'f}{H_L - m'^2 c} : \frac{n'd}{H_L - n'^2(h - d^2/f)} = m'f : n'd \quad (9)$$

in virtue of relation (7) and since $H_L \neq m'^2 c$.

Since the remaining roots of (4) are equal, the other displacement vectors are indeterminate except in so far as they lie in the plane normal to the quasi-longitudinal vector $(q_L : q_L : r_L)$ defined by (9). Hence we may express such a displacement vector in the form

$$p_T : q_T : r_T = \left[\frac{1}{\sqrt{2}}(\cos\psi + r_L \sin\psi) : \frac{1}{\sqrt{2}}(-\cos\psi + r_L \sin\psi) : -\sqrt{2}\,.\,q_L \sin\psi\right], \quad (10)$$

where ψ is an angle measured from $(1, \bar{1}, 0)$ in the plane of normal $(q_L : q_L : r_L)$.

Now, referring to Fig. 1, we may see the relation

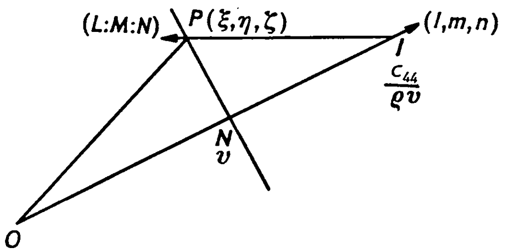

Fig. 1. The relation between corresponding velocity, inverse and wave points when $c_{44}/\varrho v > v$, as for a T-wave.

between corresponding points of the velocity, inverse and wave surfaces, and the directed line segments of the triangle NIP may be expressed in the following forms:

$$\vec{IN} = \frac{S}{2\varrho v}[l, m, n],$$

$$\vec{IP} = \frac{1}{2\varrho v}[L, M, N],$$

and

$$\vec{NP} = \frac{1}{2\varrho v}[L - lS, M - mS, N - nS],$$

where

$$S = lL + mM + nN,$$

and L, M, N are quantities whose general forms for a tetragonal medium are

$$L = (2p^2/l)(H - m^2 k) + 2q^2 lk, \quad (11)$$

$$M = 2p^2 mk + (2q^2/m)(H - l^2 k), \quad (12)$$

$$N = (2r^2/n)H. \quad (13)$$

A full derivation and discussion of these quantities has been previously published (Musgrave, 1954).

In the special case of transverse waves defined by (10) we have

$$[L_T, M_T, N_T] = 2m'[p_T^2(c-k) + q_T^2 k,\ p_T^2 k + q_T^2(c-k),\ (m'/n')r_T^2 c]. \quad (14)$$

Now all the displacement vectors represented by (10) as the angle ψ is varied are associated with a single velocity, hence we see that $\vec{IN}$ remains constant while $\vec{IP}$ and $\vec{NP}$ vary, and the wave point P traces a curve containing the extremities of all the rays corresponding to the range of displacement vectors as ψ varies from 0 to π. This curve lies in the plane of normal (m', m', n') and we may examine its character more closely by resolving $\vec{NP}$ along orthogonal axes lying in that plane.

The curve of tangency

Let us write $\vec{NP} = \mathbf{R}/2\varrho v_T$ and relate $\mathbf{R}$ to cartesian axes in directions $(1/\sqrt{2})(1, -1, 0)$ and $(1/\sqrt{2})(n', n', -2m')$ respectively.

Then

$$X = \frac{1}{\sqrt{2}}(L - M) = \sqrt{2}\,.\,m'(c - 2k)(p_T^2 - q_T^2) = \sqrt{2}\,.\,m' r_L(c - 2k)\sin 2\psi, \quad (15)$$

and

$$Y = \frac{n'}{\sqrt{2}}\left[(L + M) - \frac{2m'}{n'}N\right] = \sqrt{2}\,.\,m'n'c\left[(p_T^2 + q_T^2) - \frac{2m'^2}{n'^2}r_T^2\right] = \frac{m'n'c}{\sqrt{2}}[(1 - K) + (1 + K)\cos 2\psi], \quad (16)$$

where

$$K = (4m'^2/n'^2)q_L^2 - r_L^2,$$

and (15) and (16) are clearly parametric equations for an ellipse which is described completely as ψ varies from 0 to π. It is also evident that the centre of the ellipse is displaced in the Y direction from the origin of co-ordinates (X, Y) which is the foot of the normal from O to the plane of the wave.

As ψ varies, (p_T, q_T, r_T) rotates in the plane of normal (q_L, q_L, r_L) while the extremity of the associated ray traces the ellipse in the plane of normal (m', m', n'), the rotation of the vector $\mathbf{R}$ being in the opposite sense to that of (p_T, q_T, r_T).

The deviation Δ_T of any ray from the normal (m', m', n') is given by

$$\tan \Delta_T = R/2\varrho v_T^2\,, \tag{17}$$

and the deviation δ_L of the quasi-longitudinal displacement vector from (m', m', n') may be obtained from

$$\cos \delta_L = 2m'q_L + n'r_L\,. \tag{18}$$

It is also worth noting that when, as in the case of cubic symmetry,

$$k = 0 \quad \text{and} \quad m' = n' = 1/\sqrt{3} = q_L = r_L\,,$$

then $K = 1$ and

$$X = \tfrac{1}{3}\sqrt{2}\,.\,c \sin 2\psi, \quad Y = \tfrac{1}{3}\sqrt{2}\,.\,c \cos 2\psi\,,$$

showing that the elliptic cone of refraction degenerates to a circular cone.

The values of the elastic constants of zircon published by Bhimasenachar & Venkataratnam (1955) satisfy condition (8) and the particulars of the elliptic cone of internal refraction for this medium are given in the following section.

The elliptic cone of internal refraction for zircon

The elastic constants of zircon have been given as

$c_{11}=7{\cdot}35$, $c_{12}=0{\cdot}90$, $c_{33}=4{\cdot}60$, $c_{13}=-0{\cdot}54$, $c_{44}=1{\cdot}38$

and

$$c_{66} = 1{\cdot}60, \quad (\text{all } \times 10^{11} \text{ dyne cm.}^{-2})\,,$$

giving $a = 5{\cdot}97$, $c = 3{\cdot}69$, $d = 0{\cdot}84$, $f = 2{\cdot}5$, $h = 3{\cdot}22$ and $k = 0{\cdot}22$. The density, according to Miers, is $\varrho = 4{\cdot}7$ g.cm.$^{-3}$. Hence we find that

$$(fh-d^2)/cf = 0{\cdot}845$$

and for the double root $H_T = m'^2c$, $m' = 0{\cdot}56$ and $n' = 0{\cdot}61$, so that $v_T = 2{\cdot}32\times10^5$ cm.sec.$^{-1}$.

Also $q_L = 0{\cdot}685$ and $r_L = 0{\cdot}250$, whence $\cos \delta_L = 0{\cdot}917$ and $\delta_L = 23{\cdot}5°$ and the deviation of the L ray from (m', m', n') is $\Delta_L = 26°$.

The radius vector to the centre of the ellipse makes an angle of 5·3° with the wave normal.

Maximum deviations of ray from wave normal occur in the $(1, \bar{1}, 0)$ plane when the extremity of the ray crosses the major axis of the ellipse, that is for

$$\psi = 0 \quad \text{when} \quad (\Delta_T)_{\psi=0} = 19{\cdot}4°\,,$$
$$\psi = \tfrac{1}{2}\pi \quad \text{when} \quad (\Delta_T)_{\psi=\frac{1}{2}\pi} = 30°\,.$$

Minimum deviations occur when $\psi = \frac{1}{2}\cos^{-1} 0{\cdot}152$ and $(\Delta_T)_{\text{min.}} = 4{\cdot}6°$.

The ratio of the major to the minor axis of the ellipse is 3·49:1.

Fig. 2 illustrates the behaviour anticipated from the above analysis and computation.

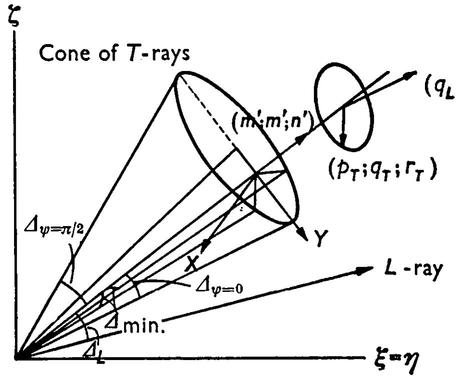

Fig. 2. The possible displacement vectors and rays for the wave normal (m', m', n') showing the elliptic cone of internal refraction.

Possibilities of experimental verification

It would be interesting to make an experimental investigation of this predicted phenomenon. Such a test would require a substantial and suitably oriented specimen of zircon when the attachment of a transversely polarized transmitter of high-frequency vibration in the same position on the crystal, but with varying directions of displacement vector, would presumably launch beams of both quasi-longitudinal and quasi-transverse waves. Since each of these waves has a displacement vector with a longitudinal component, the search for the position on the receiving face at which the energy arrives may be conducted with a transducer sensitive to longitudinal displacement. This would enable an oil film to be used as adequate acoustic contact and allow immediate mobility of the receiving transducer. In some such manner the elliptic cone of refraction could certainly be investigated if, in accordance with this analysis and the published data, it were found to exist.

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